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Integrable deformation of the Toda chain and quasigraded Lie algebras

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Abstract
We construct a family of quasigraded Lie algebras that coincide with the deformations of the loop algebras in 'principal' gradation and admit Kostant–Adler–Symes scheme. Using the constructed algebras we obtain integrable 'magnetic' deformation of the ordinary open and closed Toda chains for all series of classical matrix Lie algebras.

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1. Introduction

It is known that integrability of the majority of integrable Hamiltonian systems is based on the Lie algebras with special properties [5]. In particular, integrability of the open Toda chains [1–3] is based on the classical simple Lie algebras; integrability of the closed Toda chains [3] is based on the loop algebras in principal gradation [4, 5]. The main property of the classical simple Lie algebras and loop algebras that permit their usage in the theory of integrable systems is their property to be decomposable into direct sum of two subalgebras (the so-called Kostant–Adler–Symes scheme) [5].

In the papers [8, 9], it was shown that a special Lie algebra $\mathfrak{g}_{\mathcal{E}}$, living on an elliptic curve $\mathcal{E}$, also possesses the decomposition $\mathfrak{g}_{\mathcal{E}} = \mathfrak{g}^+_{\mathcal{E}} + \mathfrak{g}^-_{\mathcal{E}}$ and, hence, admits the Kostant–Adler–Symes scheme. In our papers [10, 12], we have generalized results of [8, 9] onto the case of special Lie algebras $\mathfrak{g}_{\mathcal{H}}$ living on the algebraic curve $\mathcal{H}$. In paper [13], we gave a Lie algebraic explanation of this construction. We have constructed a family of quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A$ possessing the decomposition $\widetilde{\mathfrak{g}}_A = \widetilde{\mathfrak{g}}^+_A + \widetilde{\mathfrak{g}}^-_A$ parametrized by some numerical matrices A, that may be viewed as quasigraded deformations of loop algebras in a homogeneous gradation, such that loop algebras themselves correspond to the case $A \equiv 0$ and quasigraded Lie algebras $\mathfrak{g}_{\mathcal{H}}$ correspond to the case $A \in \mathrm{Diag}(n)$. Using $\widetilde{\mathfrak{g}}_A$ algebras we have obtained some known and a number of new integrable Hamiltonian systems [14, 15].

Majority of the integrable Hamiltonian systems obtained in [14, 15] are top-like systems. In the present paper, we develop our approach to the integrable systems based on quasigraded Lie algebras in order to show that using our method it is possible to obtain other types of integrable systems, in particular, integrable multi-particle Toda-type systems. The last systems are very interesting by themselves, due the fact that there are only few examples of integrable multi-particle systems known (see [16] for a review). Construction of new types of such systems may be of great interest for both mathematics and physics.

In order to achieve this goal we combine our previous results [13–15], and ideas of [17] and define a new type of the quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$ admitting the Kostant–Adler–Symes scheme and coinciding with deformations of the loop algebras in the principal gradation. More definitely, it turned out that for the special choice of the matrices A (that depends on the classical matrix Lie algebras $\mathfrak{g}$) it is possible to define 'principal' subalgebras $\widetilde{\mathfrak{g}}_A^{\rm pr} \subset \widetilde{\mathfrak{g}}_A$ in the analogous way as for the case of ordinary loop algebras [4]. In the $A \to 0$ case the algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$ coincide with the ordinary loop algebras in the principal gradation.

We study properties of the Lie algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$ for all the classical matrix Lie algebras $\mathfrak{g}$. We construct their coadjoint representations and an infinite set of their invariants. Following the standard procedure [5] we introduce the 'direct-difference' Lie–Poisson bracket into linear spaces $\left(\widetilde{\mathfrak{g}_A^{\rm pr}}\right)^*$. As a result we obtain an infinite number of commuting with respect to the 'direct-difference' Lie–Poisson bracket functions on the dual spaces of our algebras. That permits us to develop the theory of integrable systems based on the Lie algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$. We concentrate our attention on the theory of finite-dimensional Hamiltonian systems connected with the algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$. In order to obtain these systems in the framework of our construction, we use the fact that the algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$ are quasigraded. This property permits us to define an infinite sequence of ideals of finite co-dimensions in the algebra $\widetilde{\mathfrak{g}}_A^{\rm pr}$ equipped with the 'direct difference' bracket. As the result, we obtain a large number of commuting functions on the dual space of each quotient algebra of a finite dimension; and hence, integrable Hamiltonian systems in the corresponding quotient spaces.

We consider in details the most interesting Hamiltonian systems in the quotient spaces of a small quasidegree. Simplest of them coincide with the integrable 'magnetic' deformations of the closed and open Toda chains. The corresponding Hamiltonians differ from the standard Toda Hamiltonians by an additional potential term and a 'magnetic' lengthening of impulses. They have for the all classical matrix Lie algebras of the rank n the following explicit form:

$$H = \frac{1}{2}\sum_{i=1}^{n}\left(p_i + \sum_{\alpha_j\in\Pi\bigcup-\Theta}\beta_{i,j}^{A}\,{\rm e}^{\alpha_j(q)}\right)^2 - \frac{1}{2}\sum_{\alpha_j,\alpha_i\in\Pi\bigcup-\Theta}\gamma_{i,j}^{A}\,{\rm e}^{\alpha_i(q)}\,{\rm e}^{\alpha_j(q)} + \sum_{\alpha_i\in\Pi\bigcup-\Theta}c_i\,{\rm e}^{\alpha_i(q)}, \tag{1}$$

where Π is a system of simple roots of $\mathfrak{g}$, Θ is the longest root, $\beta_{i,j}^A$ and $\gamma_{i,j}^A$ are the functions in the matrix elements of the 'deformation' matrix A, q belongs to the Cartan subalgebra: $q = \sum_{i=1}^n q_i H_i$ where $H_i \in \mathfrak{h}$ and p_i, q_i are the standard coordinates with the canonical bracket.

The Hamiltonian (1) has the simplest form in the case $\mathfrak{g} = gl(n)$:

$$H = \frac{1}{2}\sum_{i=1}^{n}\left(p_i + \frac{1}{2}(a_i\,{\rm e}^{q_i-q_{i+1}} + a_{i-1}\,{\rm e}^{q_{i-1}-q_i})\right)^2 - \frac{1}{2}\sum_{i=1}^{n}a_i^2\,{\rm e}^{2(q_i-q_{i+1})} + \sum_{i=1}^{n}c_i\,{\rm e}^{q_i-q_{i+1}}, \tag{2}$$

where $n+1 \equiv 1$, c_i are constants of interaction and a_i are the deformation parameters, i.e. nontrivial matrix elements of the matrix A. In the $\mathfrak{g} = gl(n)$ case and $\mathfrak{g} = sp(n)$, constants c_i and a_i are independent and arbitrary. In the $\mathfrak{g} = so(n)$ case they are subjected to some additional constraints of an algebraic origin. In the $c_{n+1} \neq 0$ case in the limit $a_i \to 0$

Hamiltonians (1) tend to the standard closed Toda chain Hamiltonians of [3]. In the $c_{n+1} = 0$ case in the limit $a_i \to 0$ Hamiltonians (1) tend to the standard open Toda chain Hamiltonians of [1].

In the present paper, we also construct Lax pairs and the 'deformed' Lax equations that are equivalent to the Hamiltonian equations of motion of our systems and spectral curves that correspond to the 'deformed' Lax equations.

The structure of the article is as follows: in section 2, we describe a principal grading of simple (reductive) Lie algebras. In section 3, we construct the corresponding 'principal' quasigraded Lie algebras. In section 4, we describe their dual spaces, coadjoint invariants, Lie–Poisson brackets, Lie–Poisson subspaces and integrable systems of the Euler–Arnold type on them. At last, in section 5, we obtain integrable deformations of the open and closed Toda chains.

2. Principal grading of simple Lie algebras

In this subsection, we will introduce necessary notations and remind some important facts from the theory of semisimple Lie algebras [4]. Let algebra $\mathfrak{g}$ with the bracket $[,]$ be simple (reductive) classical Lie algebra of the rank n. Let $\mathfrak{h} \subset \mathfrak{g}$ be its Cartan subalgebra, $\Delta_\pm$ be its set of positive(negative) roots, Π be the set of simple roots, $H_i \in \mathfrak{h}$ be the basis of Cartan subalgebra $E_\alpha, \alpha \in \Delta$ corresponding root vectors.

Let us define the so-called 'principal' grading of $\mathfrak{g}$ [4], putting:

$$\deg H_i = 0, \qquad \deg E_{\alpha_i} = 1, \qquad \deg E_{-\alpha_i} = -1.$$

It is evident that in such a way we obtain grading of $\mathfrak{g}$: $\mathfrak{g} = \sum_{k=0}^{h-1} \mathfrak{g}_{\overline{k}}$ with the graded subspaces $\mathfrak{g}_{\overline{k}}$ be defined as follows: $\mathfrak{g}_{\overline{k}} = \mathrm{Span}_C\{E_\alpha\}$, where α is the root of the length k, i.e. $\alpha = \sum_{i=1}^{r} k_i E_{\alpha_i}$ if $\alpha \in \Delta_+, \alpha = \sum_{i=1}^{r} k_i E_{-\alpha_i}$ if $\alpha \in \Delta_-$ and $k = \sum_{i=1}^{r} k_i, h$ is the Coxeter number of $\mathfrak{g}$. In particular $\mathfrak{g}_{\overline{0}} = \mathfrak{h}$, $\mathfrak{g}_{\overline{1}} = \mathrm{Span}_C\{E_{\alpha_i}, E_{-\theta} | \alpha_i \in \Pi\}$, $\mathfrak{g}_{\overline{-1}} = \mathrm{Span}_C\{E_{-\alpha_i}, E_\theta | \alpha_i \in \Pi\}$ and θ is the longest root of the length $h-1$.

Let us consider the following examples of classical matrix Lie algebras.

Example 1. Let us consider the case of $\mathfrak{g} = gl(n)$ with the basis $(X_{ij})_{ab} = \delta_{ia}\delta_{jb}, i, j \in 1, n$ and the standard commutation relations

$$[X_{ij}, X_{kl}] = \delta_{kj} X_{il} - \delta_{il} X_{kj}.$$

In this case, $\mathfrak{g}_{\overline{1}} = \mathrm{Span}_C\{E_{\alpha_i} \equiv X_{ii+1}, E_{-\theta} \equiv X_{n1} | i \in 1, n-1\}$, $\mathfrak{g}_{\overline{0}} = \mathrm{Span}_C\{H_i \equiv X_{ii} | i \in 1, n\}$, $\mathfrak{g}_{\overline{-1}} = \mathrm{Span}_C\{E_{-\alpha_i} \equiv X_{i+1,1}, E_\theta \equiv X_{1n} | i \in 1, n-1\}$ and the Coxeter number is $h = n$.

Example 2. Let us consider the case of $\mathfrak{g} = so(2n+1)$, where $\mathrm{so}(2n+1) = \{X \in gl(2n+1) | X = -sX^\top s\}$ where $s = \mathrm{diag}(1, s_{2n})$, $s_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. In such realization, the Cartan subalgebra has a basis $H_i = X_{i+1,i+1} - X_{i+n+1,i+n+1}$, where $i = 1, n$ generators of algebra that correspond to the simple roots are $E_{\alpha_i} = X_{i+1,i+2} - X_{n+i+2,n+i+1}, i = 1, n-1$, $E_{\alpha_n} = X_{n+1,1} - X_{1,2n+1}$, their negative counterparts are $E_{-\alpha_i} = X_{i+2,i+1} - X_{n+i+1,n+i+2}, i = 1, n-1$, $E_{-\alpha_n} = X_{1,1+n} - X_{2n+1,1}$. The longest root corresponds to $E_\theta = X_{32+n} - X_{23+n}$, its negative counterpart to $E_{-\theta} = X_{2+n3} - X_{3+n2}$, the Coxeter number $h = 2n$.

Example 3. Let us consider the case of $\mathfrak{g} = sp(n)$, where $sp(n) = \{X \in gl(2n) | X = wX^\top w\}$, where $w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. In such realization, the Cartan subalgebra has a basis $H_i = X_{i,i} - X_{i+n,i+n}$, where $i = 1, n$ generators of algebra that correspond to the simple roots are $E_{\alpha_i} = X_{i,i+1} - X_{n+i+1,n+i}, i = 1, n-1$, $E_{\alpha_n} = X_{n,2n}$, their negative counterparts are $E_{-\alpha_i} = X_{i+1,i} - X_{n+i,n+i+1}$, $i = \overline{1, n-1}$, $E_{-\alpha_n} = X_{2n,n}$. The longest root corresponds to $E_\theta = X_{11+n}$, its negative counterpart to $E_{-\theta} = X_{1+n1}$, the Coxeter number $h = 2n$.

Example 4. Let us consider the case of $\mathfrak{g} = so(2n)$, where $so(2n) = \{X \in gl(2n) | X = -sX^\top s\}$, where $s \equiv s_{2n})$, $s_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. In such realization, the Cartan subalgebra has a basis $H_i = X_{i,i} - X_{i+n,i+n}$, where $i = \overline{1, n}$ generators of algebra that correspond to the simple roots are $E_{\alpha_i} = X_{i,i+1} - X_{n+i+1,n+i}$, $i = \overline{1, n-1}$, $E_{\alpha_n} = X_{n,2n-1} - X_{n-1,2n}$, their negative counterparts are $E_{-\alpha_i} = X_{i+1,i} - X_{n+i,n+i+1}$, $i = \overline{1, n-1}$, $E_{-\alpha_n} = X_{2n-1,n} - X_{2n,n-1}$. The longest root corresponds to $E_\theta = X_{21+n} - X_{12+n}$, its negative counterpart to $E_{-\theta} = X_{1+n2} - X_{2+n1}$, the Coxeter number $h = 2n - 2$.

3. 'Principal' quasigraded Lie algebras

3.1. General case

It is well-known [4] that having the 'principal' grading of $\mathfrak{g}$ it is possible to define the corresponding grading of loop space. Let $\mathfrak{g} = \sum_{k=0}^{h-1} \mathfrak{g}_{\bar{k}}$ be a $\mathbb{Z}/h\mathbb{Z}$ grading of $\mathfrak{g}$. Let us consider the subspace $\widetilde{\mathfrak{g}}^{\mathrm{pr}} \subset \widetilde{\mathfrak{g}}$, where $\widetilde{\mathfrak{g}} \equiv \mathfrak{g} \otimes P(\lambda, \lambda^{-1})$ of the following type:

$$\widetilde{\mathfrak{g}}^{\mathrm{pr}} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\bar{j}} \otimes \lambda^j. \tag{3}$$

Here, $\bar{j}$ denotes a class of equivalence of the elements $j \in \mathbb{Z} \bmod h\mathbb{Z}$. From the fact that $[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subset \mathfrak{g}_{\overline{i+j}}$ it follows that $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$ is a closed Lie algebra with respect to the ordinary Lie bracket on the tensor product:

$$[X \otimes p(\lambda), Y \otimes q(\lambda)] = [X, Y] \otimes p(\lambda)q(\lambda),$$

where $X \otimes p(\lambda), Y \otimes q(\lambda) \in \widetilde{\mathfrak{g}}^{\mathrm{pr}}$. It is evident from the very definition that $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$ is the graded Lie algebra with the grading being defined by the degrees of the spectral parameter λ.

Let us introduce the structure of the quasigraded Lie algebra into the loop space $\widetilde{\mathfrak{g}}$. In order to do this we will deform Lie algebraic structure in loop algebras $\widetilde{\mathfrak{g}}$ in the following way [13–15]:

$$[X \otimes p(\lambda), Y \otimes q(\lambda)]_F = [X, Y] \otimes p(\lambda)q(\lambda) - F(X, Y) \otimes \lambda p(\lambda)q(\lambda), \tag{4}$$

where $X \otimes p(\lambda), Y \otimes q(\lambda) \in \widetilde{\mathfrak{g}}$ and the map $F : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is skew and satisfies the following two requirements which are equivalent to the Jacobi identities:

(J1) $\sum_{c.p.\{i,j,k\}} (F([X_i, X_j], X_k) + [F(X_i, X_j), X_k]) = 0$,
(J2) $\sum_{c.p.\{i,j,k\}} F(F(X_i, X_j), X_k) = 0$.

Now we are interested in the possibility of defining the structure of the quasigraded algebra on the space $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$. For this purpose, we want bracket (4) to be correctly restricted to the space $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$. By the direct verification one can prove the following proposition.

Proposition 3.1. *The subspace $\widetilde{\mathfrak{g}}^{\mathrm{pr}} \subset \widetilde{\mathfrak{g}}$ is the closed Lie algebra if and only if:*

$$F(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) \subset \mathfrak{g}_{\overline{i+j+1}}. \tag{5}$$

In the next subsection, we will explicitly present examples of the cocycles F on the finite-dimensional Lie algebras $\mathfrak{g}$ that satisfies conditions (J1), (J2) and (5).

3.2. Case of classical matrix Lie algebras

Let us now consider the classical matrix Lie algebras $\mathfrak{g}$ of the type $gl(n)$, $so(n)$ and $sp(n)$ over the field $\mathbb{K}$ of the complex or real numbers. As in the examples above, we will realize the algebra $so(n)$ as algebra of skew-symmetric matrices: $so(n) = \{X \in gl(n) | X = -sX^\top s\}$ and algebra $sp(n)$ as the following matrix Lie algebra: $sp(n) = \{X \in gl(2n) | X = wX^\top w\}$, where $s \in \mathrm{symm}(n)$, $s^2 = 1$, $w \in so(2n)$ and $w^2 = -1$.

Let us consider the cochain $F : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of the following explicit form:

$$F_A(X, Y) = [X, Y]_A \equiv XAY - YAX.$$

From the theory of consistent Poisson brackets it is known [19] to satisfy conditions (J1) and (J2).

The following proposition holds true:

Proposition 3.2. *The cochain F_A satisfies condition (5) if and only if the matrix A has the form:*

(i) $A = \sum_{i=1}^{n-1} a_i X_{ii+1} + a_n X_{n1}$ *if* $\mathfrak{g} = gl(n)$,
(ii) $A = \sum_{i=1}^{n-1} a_i (X_{i+1,i+2} + X_{n+i+2,n+i+1}) + a_n (X_{1+n,1} + X_{1,2n+1}) + a_{n+1} (X_{2+n3} + X_{3+n2})$ *if* $\mathfrak{g} = so(2n+1)$,
(iii) $A = \sum_{i=1}^{n-1} a_i (X_{i,i+1} + X_{n+i+1,n+i})$ *if* $\mathfrak{g} = sp(n)$,
(iv) $A = \sum_{i=1}^{n-1} a_i (X_{i,i+1} + X_{n+i+1,n+i}) + a_n (X_{n,2n-1} + X_{n-1,2n}) + a_{n+1} (X_{1+n2} + X_{2+n1})$ *if* $\mathfrak{g} = so(2n)$,

where X_{ij} is a standard matrix basis of $gl(n)$, $(X_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{\gamma\beta}$.

Proof. For the bracket constructed with the help of the cocycle F_A to be correctly defined on $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$ we require that linear space $\mathfrak{g}$ be closed with respect to the bracket $[,]_A$ and linear space $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$ as a space of matrix-valued function of λ be closed with respect to the bracket (6). These conditions are equivalent to the following requirement: $[X, Y]_A \in \mathfrak{g}_{i+j+1} \forall X \in \mathfrak{g}_i, Y \in \mathfrak{g}_j$. Straightforward case-by-case verification shows that this requirement is satisfied if and only if the matrix A has the form described in the proposition. □

Hence in the case of the matrix Lie algebras and the matrices A defined in the above proposition we may introduce into the space $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$ the new Lie bracket of the form:

$$[X \otimes p(\lambda), Y \otimes q(\lambda)] = [X, Y] \otimes p(\lambda)q(\lambda) - [X, Y]_A \otimes \lambda p(\lambda)q(\lambda), \tag{6}$$

where $X \otimes p(\lambda), Y \otimes q(\lambda) \in \widetilde{\mathfrak{g}}^{\mathrm{pr}}$, $[X, Y] \equiv XY - YX$ in the right-hand side of this identity denote an ordinary Lie bracket in $\mathfrak{g}$ and $[X, Y]_A \equiv XAY - YAX$.

Definition. *We will denote the linear space $\widetilde{\mathfrak{g}}^{\mathrm{pr}}$ with the bracket given by (6) by $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$.*

From the very definition of $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ it follows that the algebra $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ is Z-quasigraded and $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ admits the direct sum decomposition $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A = \widetilde{\mathfrak{g}}^{pr+}_A + \widetilde{\mathfrak{g}}^{pr-}_A$, where

$$\widetilde{\mathfrak{g}}^{pr+}_A = \bigoplus_{j\geqslant 0} \mathfrak{g}_{\bar{j}} \otimes \lambda^j, \qquad \widetilde{\mathfrak{g}}^{pr-}_A = \bigoplus_{j<0} \mathfrak{g}_{\bar{j}} \otimes \lambda^j. \tag{7}$$

4. Dual space, Poisson bracket and algebra of integrals

In order to describe applications of the Lie algebras $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ to the theory of finite-dimensional integrable Hamiltonian systems, it is necessary to define the linear space $\left(\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A\right)^*$, the corresponding Lie–Poisson bracket and the Casimir functions of $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$.

4.1. Coadjoint representation and invariant functions of $\widetilde{\mathfrak{g}}_A^{\rm pr}$

In this subsection, we will construct the dual space, coadjoint representation and its invariants for the case of the 'principal' quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A^{\rm pr}$. If h is the Coxeter number then from the properties of invariant form on simple Lie algebras it follows [4] that $(\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}) = 0$ if $i + j \neq 0 \bmod h$. Hence we can define a pairing between $\widetilde{\mathfrak{g}}_A^{\rm pr}$ and $\left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*$ in the following way:

$$\langle X, L\rangle = \mathrm{res}_{\lambda=0}\lambda^{-1}\,\mathrm{Tr}(X(\lambda)L(\lambda)). \tag{8}$$

From this definition it follows that the generic element $L(\lambda) \in \left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*$ has the form:

$$L(\lambda) = \sum_{j\in Z}\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{j}}} l_\alpha^{(j)} X_\alpha^{\overline{-j}}\lambda^{-j}, \tag{9}$$

where $X_\alpha^{\overline{-j}}$ is a basis element of subspace $\mathfrak{g}_{\overline{-j}}$ and $l_\alpha^{(j)}$ is a coordinate function on $\left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*$.

The following proposition holds true.

Proposition 4.1. *Let $L(\lambda) \in \left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*$ be the generic element of the dual space. Then the functions,*

$$I_k^m(L(\lambda)) = \frac{1}{m}\mathrm{res}_{\lambda=0}\lambda^{-k-1}\,\mathrm{Tr}(L(\lambda)A(\lambda)^{-1})^m, \tag{10}$$

are invariants of the coadjoint representation of the Lie algebra $\widetilde{\mathfrak{g}}_A^{\rm pr}$

Proof. It follows from the explicit form of the coadjoint action which has the following form:

$$ad^*_{X(\lambda)} \circ L(\lambda) = A(\lambda)X(\lambda)L(\lambda) - L(\lambda)X(\lambda)A(\lambda), \tag{11}$$

where $A(\lambda) = (1 - \lambda A)$, $X(\lambda), Y(\lambda) \in \widetilde{\mathfrak{g}}_A^{\rm pr}$, $L(\lambda) \in \left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*$. □

Remark 1. Matrix $A(\lambda)^{-1} \equiv (1 - \lambda A)^{-1}$ in the above proposition has to be understood as a power series in λ in the neighborhood of 0 or ∞: $A(\lambda)^{-1} = (1 + A\lambda + A^2\lambda^2 + \cdots)$ or $A(\lambda)^{-1} = -(A^{-1}\lambda^{-1} + A^{-2}\lambda^{-2} + \cdots)$.

4.2. Two Lie–Poisson brackets

Let us define Poisson structures in the space $\left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*$. Using pairing (8) described in the previous section we can define the Lie–Poisson bracket on $P\left(\left(\widetilde{\mathfrak{g}}_A^{\rm pr}\right)^*\right)$ in the standard way:

$$\{F_1(L(\lambda)), F_2(L(\lambda))\} = \langle L(\lambda), [\nabla F_1(L(\lambda)), \nabla F_2(L(\lambda))]_{A(\lambda)}\rangle, \tag{12}$$

where $\nabla F_i(L(\lambda)) = \sum_{j\in Z}\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{j}}} \frac{\partial F_i}{\partial l_\alpha^{(j)}} X_\alpha^{\bar{j}}\lambda^j$ and $X_\alpha^{\bar{j}}$ is a basis element of subspace $\mathfrak{g}_{\bar{j}}$.

From proposition 4.1 and standard considerations follows the next corollary:

Corollary 4.1. *Functions $I_k^m(L(\lambda))$ are central (Casimir) functions for the Lie–Poisson bracket (12).*

Let us calculate Poisson bracket (12) explicitly. It is easy to show that for the coordinate functions $l_\alpha^{(i)}, l_\beta^{(j)}$, where $l_\alpha^{(i)} \in (\mathfrak{g}_{\bar{i}})^*$, $l_\beta^{(j)} \in (\mathfrak{g}_{\bar{j}})^*$, this bracket will have the following form:

$$\left\{l_\alpha^{(i)}, l_\beta^{(j)}\right\} = \sum_\gamma C_{\alpha,\beta}^\gamma l_\gamma^{(i+j)} - \sum_\delta C_{\alpha,\beta}^\delta(A) l_\delta^{(i+j+1)}, \tag{13}$$

where l_γ and l_δ are the coordinate functions on $(\mathfrak{g}_{\overline{i+j}})^*$ and $(\mathfrak{g}_{\overline{i+j+1}})^*$.

Let us now introduce into the space $\left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^*$ a new Poisson bracket $\{,\}_0$, which is a Lie–Poisson bracket for the algebra $\left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^0$, where $\left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^0 = \left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^- \ominus \left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^+$. Explicitly, this bracket has the following form:

$$\left\{l_\alpha^{(i)}, l_\beta^{(j)}\right\}_0 = \left\{l_\alpha^{(i)}, l_\beta^{(j)}\right\}, \quad i, j \geqslant 0, \qquad \left\{l_\alpha^{(i)}, l_\beta^{(j)}\right\}_0 = -\left\{l_\alpha^{(i)}, l_\beta^{(j)}\right\}, \quad i, j < 0, \tag{14}$$

$$\left\{l_\alpha^{(i)}, l_\beta^{(j)}\right\}_0 = 0, \qquad j < 0, \quad i \geqslant 0 \text{ or } i < 0, \quad j \geqslant 0. \tag{15}$$

4.3. *Finite-dimensional Poisson subspaces $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$*

In order to obtain the finite-dimensional integrable systems in the framework of the above construction one should be able to restrict the Poisson bracket $\{\,,\}_0$ on some finite-dimensional subspaces of $\left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^*$.

Let the finite-dimensional subspace $\mathcal{M}_{s,p}^{\mathrm{pr}}(A) \subset \left(\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}\right)^*$, $s, p \geqslant 0$, be defined as follows:

$$\mathcal{M}_{s,p}^{\mathrm{pr}}(A) = \sum_{m=-s}^{p-1} \left(\widetilde{\mathfrak{g}}_m^{\mathrm{pr}}\right)^*.$$

The following proposition is true:

Proposition 4.2. *Bracket $\{\,,\}_0$ is correctly restricted to $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$.*

Proof. It follows from the fact that the subspaces $\left(\mathcal{J}_{p,s}^{\mathrm{pr}}\right)^* = \sum_{m=-\infty}^{-s-1} \left(\widetilde{\mathfrak{g}}_m^{\mathrm{pr}}\right)^* + \sum_{m=p}^{\infty} \left(\widetilde{\mathfrak{g}}_m^{\mathrm{pr}}\right)^*$ are ideals in the Poisson algebra defined by the bracket $\{,\}_0$ and restriction of the bracket $\{\,,\}_0$ on $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$ is equivalent to factorization of Poisson algebras over these ideals. □

The Lax operator $L(\lambda)$ in the subspace $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$ has the form:

$$L(\lambda) = \sum_{j=-s}^{p-1} L^{(j)}\lambda^{-j} = \sum_{j=-s}^{p-1} \sum_{\alpha=1}^{\dim\mathfrak{g}_{\overline{-j}}} l_\alpha^{(j)}\lambda^{-j} X_\alpha^{\overline{-j}}.$$

where $L^{(k)} \in \mathfrak{g}_{\overline{-k}}$. The Lie–Poisson bracket on $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$ is written as follows:

$$\left\{l_\alpha^{(n)}, l_\beta^{(m)}\right\}_0 = \sum_\gamma C_{\alpha,\beta}^\gamma l_\gamma^{(n+m)} - \sum_\delta C_{\alpha,\beta}^\delta(A) l_\delta^{(n+m+1)}, \qquad \text{when} \quad n, m \geqslant 0, \quad n+m+1 < p,$$

$$\left\{l_\alpha^{(n)}, l_\beta^{(m)}\right\}_0 = -\sum_\gamma C_{\alpha,\beta}^\gamma l_\gamma^{(n+m)} + \sum_\delta C_{\alpha,\beta}^\delta(A) l_\delta^{(n+m+1)}, \qquad \text{when} \quad n, m < 0, \quad n+m+1 > -s,$$

$\left\{l_\alpha^{(n)}, l_\beta^{(m)}\right\}_0 = 0$ in other cases.

4.4. *'Deformed' Lax equations*

Let us now consider Hamiltonian equations on the Poisson subspace $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$. Let $L(\lambda) \in \mathcal{M}_{s,p}^{\mathrm{pr}}(A)$ and $H(L(\lambda))$ be the restriction of some function H on $\left(\mathfrak{g}_A^{\mathrm{pr}}\right)^*$ onto $\mathcal{M}_{s,p}^{\mathrm{pr}}(A)$. We can write the corresponding Hamiltonian equations of motion in the form:

$$\frac{\mathrm{d}l_\alpha^{(j)}(\lambda)}{\mathrm{d}t} = \left\{l_\alpha^{(j)}(\lambda), H(L(\lambda))\right\}_0, \tag{16}$$

where $\alpha \in \overline{1, \dim\mathfrak{g}_{\overline{j}}}$.

The following theorem holds true:

Theorem 4.1. *Let the Hamiltonian H be an invariant of the coadjoint representation of the Lie algebra $\widetilde{\mathfrak{g}}_A^{\mathrm{pr}}$. Then*

(i) *equations (16) can be written in the form of the 'deformed' Lax equations:*

$$\frac{\mathrm{d}L(\lambda)}{\mathrm{d}t} = A(\lambda)M^{\pm}(\lambda)L(\lambda) - L(\lambda)M^{\pm}(\lambda)A(\lambda), \tag{17}$$

where $L(\lambda) \in \mathcal{M}^{\mathrm{pr}}_{s,p}(A)$, $M^{\pm}(\lambda) = \mp(P_{\pm}\nabla H)|_{L(\lambda)\in\mathcal{M}^{\mathrm{pr}}_{s,p}(A)}$, $P_{\pm}$ *are the projection operators on the subalgebra* $\left(\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A\right)^{\pm}$, $\nabla H = \sum_{j\in\mathbb{Z}}\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{j}}} \frac{\partial H}{\partial l^{(j)}_{\alpha}} X^{j}_{\alpha}$.

(ii) *The polynomial functions* $\left\{I^r_n(L(\lambda))\right\}$, *where* $L(\lambda) \in \mathcal{M}^{\mathrm{pr}}_{s,p}(A)$ *constitute commutative (with respect to the restriction of the bracket* $\{,\}_0$ *on* $\mathcal{M}^{\mathrm{pr}}_{s,p}(A)$*) algebra of integrals of the 'deformed' Lax equations (17).*

Idea of the Proof. Item (i) of the theorem follows from the Kostant–Adler–Symes scheme [5] and explicit form of the coadjoint action of $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ (11). Item (ii) of the theorem follows from the Kostant–Adler–Symes scheme and the fact that projection onto the quotient space $\mathcal{M}^{\mathrm{pr}}_{s,p}(A)$ is a canonical homomorphism.

Remark 2. Note that the restriction of the Hamiltonian H onto the finite-dimensional subspace $\mathcal{M}^{\mathrm{pr}}_{s,p}(A)$ in the definition of the M-operator is made after the matrix gradient of H was taken.

Remark 3. It is possible to transform 'deformed' Lax equations (17) to the form of the usual Lax equations using other realizations of $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ and $\left(\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A\right)^*$. The form of the $L - M$ pairs in the last case will be more complicated. That is why we consider the realization of $\widetilde{\mathfrak{g}}^{\mathrm{pr}}_A$ presented in this paper to be the most convenient.

Now let us consider explicit form of the integrals I^r_n on the general finite-dimensional subspace $\mathcal{M}_{s,p}(A)$. Calculating these Hamiltonians, we will decompose $A^{-1}(\lambda)$ in the power series in the neighbourhood of zero:

$$I^r(\lambda) \equiv 1/r\,\mathrm{tr}(L(\lambda)(1 + A\lambda + A^2\lambda^2 + \cdots))^r \equiv \sum_{n=-rp}^{\infty} I^r_n\lambda^n. \tag{18}$$

It is not difficult to obtain the following formulae for the integrals I^r_n:

$$I^r_n = 1/r \sum_{k_1,k_2,\ldots,k_r=0}^{k_1+\cdots+k_r=n-(m_1+\cdots+m_r)} \sum_{m_1,m_2,\ldots,m_r=-p}^{s-1} \mathrm{tr}(L^{(-m_1)}A^{k_1}L^{(-m_2)}A^{k_2}\cdots L^{(-m_r)}A^{k_r}). \tag{19}$$

Standard considerations show that these integrals are not identically equal to zero only in the case when n is proportional to the Coxeter number of $\mathfrak{g}$. They generate Hamiltonian flows that, due to the results of the previous section, can be written in the Lax-type form with either of the following M-operators:

$$M^{r\pm}_k = \mp P_{\pm}(\lambda^{-k}((1 + A\lambda + A^2\lambda^2 + \cdots)L(\lambda))^{r-1}(1 + A\lambda + A^2\lambda^2 + \cdots)). \tag{20}$$

Remark 4. Note that as in classical 'non-deformed' case ([5]), we have two M-operators, M^+ and M^-, for the same Hamiltonian and Lax equations. Nevertheless, because of the multipliers $(1 + A\lambda + A^2\lambda^2 + \cdots)$ in their definition, only one of these operators has form simple enough for the usage.

5. 'Magnetic' deformation of the Toda chains

5.1. Poisson structure of the quotient space $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$

Let us discuss the Poisson structure in the quotient spaces $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$. Note that it is possible to consider the spaces $\mathcal{M}^{\mathrm{pr}}_{1,0}(A)$ and $\mathcal{M}^{\mathrm{pr}}_{0,2}(A)$ separately due to the fact that $\mathcal{M}^{\mathrm{pr}}_{1,2}(A) = \mathcal{M}^{\mathrm{pr}}_{1,0}(A) \ominus \mathcal{M}^{\mathrm{pr}}_{0,2}(A)$ both as linear and as a Poisson space.

The generic element of the space $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$ has the following form:

$$L(\lambda) = \lambda L^{(-1)} + L^{(0)} + \lambda^{-1} L^{(1)},$$

where

$$L^{(0)} = \sum_{i=1}^{n} l_i^{(0)} H_i, \qquad L^{(-1)} = \sum_{\alpha_i \in \Pi \bigcup -\Theta} l_i^{(-1)} E_{\alpha_i}, \qquad L^{(1)} = \sum_{\alpha_i \in \Pi \bigcup -\Theta} l_i^{(1)} E_{-\alpha_i}.$$

(1) Let us at first consider the space $\mathcal{M}^{\mathrm{pr}}_{0,2}(A)$. It coincides with $(\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}})^*$ as a linear space. Its generic element has the form: $L^{(0)} + \lambda^{-1} L^{(1)}$. The corresponding commutation relations are as follows:

$$\left\{l_\alpha^{(0)}, l_\beta^{(0)}\right\}_0 = \sum_\delta C^\delta_{\alpha,\beta} l_\delta^{(0)} - \sum_\delta C^\delta_{\alpha,\beta}(A) l_\delta^{(1)}, \tag{21a}$$

$$\left\{l_\alpha^{(0)}, l_\beta^{(1)}\right\}_0 = \sum_\delta C^\delta_{\alpha,\beta} l_\delta^{(1)}, \tag{21b}$$

$$\left\{l_\alpha^{(1)}, l_\beta^{(1)}\right\}_0 = 0. \tag{21c}$$

Let us show that these brackets are equivalent to the canonical Poisson brackets.

By the direct calculations one can prove the following proposition:

Proposition 5.1. *Let matrices P and Q be defined as follows: $P = (L^{(0)} + 1/2(L^{(1)}A + AL^{(1)}))$, $Q = L^{(1)}$. Then $P = \sum_{k=1}^{rk\mathfrak{g}} p_k H_k$, $Q = \sum_{\alpha_k \in \Pi \bigcup -\Theta} \mathrm{e}^{\alpha_i(q)} E_{-\alpha_k}$, where $q = \sum_{k=1}^{rk\mathfrak{g}} q_k H_k$ and brackets among p_i and q_i are canonical:*

$$\{p_i, p_i\}_0 = 0, \qquad \{p_i, q_j\}_0 = \delta_{ij}, \qquad \{q_i, q_j\}_0 = 0. \tag{22}$$

(2) Let us now consider the space $\mathcal{M}^{\mathrm{pr}}_{1,0}(A)$. It coincides with $(\mathfrak{g}_A)^*_{\bar{0}}$ both as the linear and the Poisson space. Its generic element has the form: $\lambda L^{(-1)}$. Corresponding commutation relations are as follows:

$$\left\{l_\alpha^{(-1)}, l_\beta^{(-1)}\right\}_0 = -\sum_\delta C^\delta_{\alpha,\beta}(A) l_\delta^{(-1)}.$$

It turns out that this bracket is almost always trivial. The following proposition holds true.

Proposition 5.2. *Let the deformation matrix A has the form described in proposition 3.2. Then,*

1. *$(\mathfrak{g}_A)_{\bar{0}}$ is Abelian for the all values of the parameters a_i if $\mathfrak{g} = gl(n), sp(n)$.*
2. *$(\mathfrak{g}_A)_{\bar{0}}$ is Abelian if $a_1 = a_{n+1} = 0$ and $\mathfrak{g} = so(2n+1)$.*
3. *$(\mathfrak{g}_A)_{\bar{0}}$ is Abelian if $a_1 = a_{n+1} = a_n = a_{n-1} = 0$ for $\mathfrak{g} = so(2n)$.*

Remark 5. Under the conditions of the above proposition the Abelian subspace $\mathcal{M}^{\mathrm{pr}}_{1,0}(A)$ becomes the centre of $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$, i.e. the coordinates of the subspace $\mathcal{M}^{\mathrm{pr}}_{1,0}(A)$ may be put equal to constants. Let us also note that the centre of $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$ may be made nontrivial even if the conditions of the above proposition are not satisfied, but $\mathcal{M}^{\mathrm{pr}}_{1,0}(A)$ is factorized over some ideal (this is equivalent to putting of some of the coordinate functions on $\mathcal{M}^{\mathrm{pr}}_{1,0}(A)$ to be equal to zero) such that the corresponding quotient is again the Abelian. We will consider this procedure in details in the next subsections for each classical matrix Lie algebra.

5.2. Hamiltonian and Lax pair of the deformed Toda chain

In this subsection, we obtain the integrable deformation of the standard Toda chains. It coincides with the integrable Hamiltonian system described in theorem 4.1 for the case of the Poisson space $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$. We calculate the Hamiltonian and the corresponding Lax pair for the deformed Toda chain explicitly.

(1) Let us consider the set of mutually commuting integrals $I^r(L(\lambda))$ described in the previous sections with $L(\lambda) \in \mathcal{M}^{\mathrm{pr}}_{1,2}(A)$. We will be especially interested in the second-order integrals $I^2(L(\lambda))$ which we will call the 'Hamiltonians'. The simplest function of this set is

$$I^2_0(L(\lambda)) = \tfrac{1}{2}\,\mathrm{res}_{\lambda=0}\lambda^{-1} I^2_0(L(\lambda)).$$

A direct calculation gives:

$$I^2_0(L(\lambda)) = \tfrac{1}{2}\,\mathrm{Tr}(L^{(0)} + (AL^{(1)} + L^{(1)}A))^2 - \tfrac{1}{2}\,\mathrm{Tr}(AL^{(1)})^2 + \mathrm{Tr}\,L^{(1)}L^{(-1)}.$$

As it follows from the results of the previous section a replacement of variables $L^{(0)} \equiv (P + 1/2(QA + AQ))$, $L^{(1)} \equiv Q$, $L^{(-1)} \equiv C$, where

$$P = \sum_{k=1}^{rk\mathfrak{g}} p_k H_k, \qquad Q = \sum_{\alpha_k \in \Pi \bigcup -\Theta} e^{\alpha_k(q)} E_{-\alpha_k}, \qquad C = \sum_{\alpha_k \in \Pi \bigcup -\Theta} c_k E_{\alpha_k}$$

transforms the bracket on $\mathcal{M}^{\mathrm{pr}}_{0,2}(A)$ to the canonical form. In such coordinates the Hamiltonian $H \equiv I^2_0(L(\lambda))$ acquires the following form:

$$H = \tfrac{1}{2}\,\mathrm{Tr}\big(P + \tfrac{1}{2}(AQ + QA)\big)^2 - \tfrac{1}{2}\,\mathrm{Tr}(AQ)^2 + \mathrm{Tr}\,QC, \tag{23}$$

By a direct calculation it is easy to show that this Hamiltonian has for all the classical matrix Lie algebras the form:

$$H = \frac{1}{2}\sum_{i=1}^{rk\mathfrak{g}} \left(p_i + \sum_{\alpha_j \in \Pi \bigcup -\Theta} \beta^A_{i,j}\, \mathrm{e}^{\alpha_j(q)} \right)^2 - \frac{1}{2} \sum_{\alpha_j,\alpha_i \in \Pi \bigcup -\Theta} \gamma^A_{i,j}\, \mathrm{e}^{\alpha_i(q)}\, \mathrm{e}^{\alpha_j(q)} + \sum_{\alpha_i \in \Pi \bigcup -\Theta} c_i\, \mathrm{e}^{\alpha_i(q)}, \tag{24}$$

where Π is a system of simple roots of $\mathfrak{g}$, Θ is the longest root, $\beta^A_{i,j}$ and $\gamma^A_{i,j}$ are the functions in the matrix elements of the 'deformation' matrix A of the first- and second-order, respectively. We will call this Hamiltonian to be *the integrable magnetic deformation of the Toda Hamiltonian*, and the corresponding dynamical system—*the integrable magnetic deformation of the Toda chain*. In the $A = 0$ case, it coincides with the ordinary Toda chain.

(2) As it follows from theorem 4.1 the generic element of the space $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$:

$$L(\lambda) = \lambda L^{(-1)} + L^{(0)} + \lambda^{-1} L^{(1)},$$

where $L^{(0)} = (P - 1/2(AQ + QA))$, $L^{(1)} = Q$, $L^{(-1)} = C$ coincide with the Lax operator for the Lax pair of the 'deformed' Lax equations:

$$\frac{\mathrm{d}L(\lambda)}{\mathrm{d}t} = A(\lambda)M(\lambda)L(\lambda) - L(\lambda)M(\lambda)A(\lambda).$$

Let us now consider the M-operator that corresponds to $H \equiv I_0^2$. Using formula (20), we obtain the following simple expression:

$$M(\lambda) \equiv M_0^{2-}(\lambda) = \lambda^{-1}L^{(1)} \equiv \lambda^{-1}Q.$$

The second candidate for the role of the M-operator, namely M_0^{2+} has complicated form, is useless for practical applications and we will not consider it here.

The 'deformed' Lax equations corresponding to the Hamiltonian I_0^{2+} are written as follows:

$$\frac{\mathrm{d}L^{(1)}}{\mathrm{d}t} = [L^{(1)}, L^{(0)}] - (A(L^{(1)})^2 - (L^{(1)})^2 A), \tag{25a}$$

$$\frac{\mathrm{d}L^{(0)}}{\mathrm{d}t} = [L^{(1)}, L^{(-1)}] - (AL^{(1)}L^{(0)} - L^{(0)}L^{(1)}A), \tag{25b}$$

Remark 6. There is one more equation that follows from the 'deformed' Lax equations namely

$$\frac{\mathrm{d}L^{(-1)}}{\mathrm{d}t} = -(AL^{(1)}L^{(-1)} - L^{(-1)}L^{(1)}A).$$

But, using the fact that for the chosen matrices A we have $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]_A = 0$. It is easy to prove that $\frac{\mathrm{d}L^{(-1)}}{\mathrm{d}t} = 0$, i.e. that $L^{(-1)}$ is indeed a constant matrix and the last equation is satisfied automatically.

Now, in order to obtain the Hamiltonians described in this subsection in a more explicit form, we will consider the cases of all the classical matrix Lie algebras separately.

5.3. Case $\mathfrak{g} = gl(n)$

In this case we have that the generic element of the space $\mathcal{M}_{1,2}^{\mathrm{pr}}(A)$ is written as follows:

$$L(\lambda) = \lambda\left(\sum_{i=1}^{n-1} l_i^{(-1)} X_{ii+1} + l_n^{(-1)} X_{n1}\right) + \left(\sum_{i=1}^{n} l_i^{(0)} X_{ii}\right) + \lambda^{-1}\left(\sum_{i=1}^{n-1} l_i^{(1)} X_{i+1i} + l_n^{(1)} X_{1n}\right);$$

and $A = \sum_{i=1}^{n-1} a_i X_{ii+1} + a_n X_{n1}$. The Poisson bracket among the coordinate functions has the form:

$$\{l_i^{(0)}, l_j^{(0)}\}_0 = -(\delta_{ji+1}a_i l_{ii+1}^{(1)} - \delta_{ji-1}a_j l_{jj+1}^{(1)}), \qquad \{l_i^{(0)}, l_j^{(1)}\}_0 = (\delta_{ij} - \delta_{ij+1})l_j^{(1)},$$
$$\{l_i^{(1)}, l_j^{(1)}\}_0 = 0, \qquad \{l_i^{(-1)}, l_j^{(-1)}\}_0 = \{l_i^{(-1)}, l_j^{(0)}\}_0 = \{l_i^{(-1)}, l_j^{(1)}\}_0 = 0.$$

We see that in this case the coordinate functions $l_i^{(-1)}$ are, indeed, central and we may put them to be equal to constants: $l_i^{(-1)} \equiv c_i$. Making the following replacement of variables:

$$l_i^{(1)} = \mathrm{e}^{q_i - q_{i+1}}, \qquad l_i^{(0)} = p_i - \tfrac{1}{2}(a_i\,\mathrm{e}^{q_i - q_{i+1}} + a_{i-1}\,\mathrm{e}^{q_{i-1} - q_i}),$$

where $n + 1 \equiv 1$ and we obtain canonical bracket for the coordinates p_i, q_j:

$$\{p_i, q_j\}_0 = \delta_{ij}, \qquad \{p_i, p_j\}_0 = \{q_i, q_j\}_0 = 0.$$

In these canonical coordinates the Hamiltonian (23) acquires the following explicit form:

$$H = \frac{1}{2}\sum_{i=1}^{n}\left(p_i + \frac{1}{2}(a_i\,\mathrm{e}^{q_i - q_{i+1}} + a_{i-1}\,\mathrm{e}^{q_{i-1} - q_i})\right)^2 - \frac{1}{2}\sum_{i=1}^{n} a_i^2\,\mathrm{e}^{2(q_i - q_{i+1})} + \sum_{i=1}^{n} c_i\,\mathrm{e}^{q_i - q_{i+1}}. \tag{26}$$

This Hamiltonian is an integrable 'magnetic' deformation of the Hamiltonian of the closed Toda chain. If we put $c_n = a_n = 0$, we obtain the magnetic deformation of the Hamiltonian of the open Toda chain.

The simplest integral that commutes with the H function is I_0^1. Direct calculation shows that it coincides with the total momentum $I_0^1 = \sum_{i=1}^n p_i$. Preservation of the total momentum follows also from the fact that the Hamiltonian depends on the difference of the coordinates $q_i - q_{i+1}$.

5.4. Case $\mathfrak{g} = so(2n)$

In this case, we have the generic element of the space $\mathcal{M}_{1,2}^{\mathrm{pr}}(A)$ and also have the form

$$L(\lambda) = \lambda L^{(-1)} + L^{(0)} + \lambda^{-1} L^{(1)},$$

where

$$L^{(0)} = \sum_{i=1}^{n} l_i^{(0)} (X_{ii} - X_{i+ni+n}),$$

$$L^{(-1)} = \sum_{i=1}^{n-1} l_i^{(-1)} (X_{ii+1} - X_{i+n+1i+n}) + l_n^{(-1)} (X_{n2n-1} - X_{n-12n}) + l_{n+1}^{(-1)} (X_{n+12} - X_{n+21}),$$

$$L^{(1)} = \left(\sum_{i=1}^{n-1} l_i^{(1)} (X_{i+1i} - X_{i+ni+n+1}) + l_n^{(1)} (X_{2n-1n} - X_{2nn-1}) + l_{n+1}^{(1)} (X_{2n+1} - X_{1n+2}) \right).$$

The Poisson bracket among the coordinate functions is written as follows:

$$\begin{aligned}
\{l_i^{(0)}, l_j^{(0)}\}_0 &= -\big(\delta_{ji+1} a_i l_{ii+1}^{(1)} - \delta_{ji-1} a_j l_{jj+1}^{(1)}\big) + a_n (\delta_{in}\delta_{jn-1} - \delta_{in-1}\delta_{jn}) l_n^{(1)} \\
&\quad + a_{n+1} (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}) l_{n+1}^{(1)}, \\
\{l_i^{(0)}, l_j^{(1)}\}_0 &= (\delta_{ij} - \delta_{ij+1}) l_j^{(1)}, \qquad \{l_i^{(0)}, l_n^{(1)}\}_0 = (\delta_{in} + \delta_{in-1}) l_n^{(1)}, \\
\{l_i^{(0)}, l_{n+1}^{(1)}\}_0 &= -(\delta_{i2} + \delta_{i1}) l_{n+1}^{(1)}, \qquad \{l_i^{(1)}, l_j^{(1)}\}_0 = 0, \\
\{l_i^{(-1)}, l_j^{(-1)}\}_0 &= (\delta_{jn}\delta_{in-1} - \delta_{in-1}\delta_{jn}) \big(a_{n-1} l_n^{(-1)} - a_n l_{n-1}^{(-1)}\big) \\
&\quad + (\delta_{jn+1}\delta_{i1} - \delta_{in+1}\delta_{j1}) \big(a_1 l_{n+1}^{(-1)} - a_{n+1} l_1^{(-1)}\big) \\
\{l_i^{(-1)}, l_j^{(0)}\}_0 &= \{l_i^{(-1)}, l_j^{(1)}\}_0 = 0.
\end{aligned}$$

It follows from this that the variables $l_i^{(-1)}$ are central and could be put equal to constants if and only if $a_1 l_{n+1}^{(-1)} = a_1 l_{n+1}^{(-1)} = a_{n-1} l_n^{(-1)} = a_n l_{n-1}^{(-1)} = 0$. Let us hereafter take such a matrix A and 'shift-matrix' $L^{(-1)}$ that this condition is satisfied. In this case we put $l_i^{(-1)} \equiv c_i$ and taking into account the explicit form of simple roots for $so(2n)$, introduce the described standard variables in subsection 5.1:

$$\begin{aligned}
l_i^{(1)} &= \mathrm{e}^{q_i - q_{i+1}}, \quad i = 1, n-1, \qquad l_n^{(1)} = \mathrm{e}^{q_n + q_{n-1}}, \qquad l_{n+1}^{(1)} = \mathrm{e}^{-(q_1 + q_2)}, \\
l_i^{(0)} &= p_i - \tfrac{1}{2}(a_i\, \mathrm{e}^{q_i - q_{i+1}} + a_{i-1}\, \mathrm{e}^{q_{i-1} - q_i}), \qquad i = 3, n-2, \\
l_1^{(0)} &= p_1 - \tfrac{1}{2}(a_1\, \mathrm{e}^{q_1 - q_2} - a_{n+1}\, \mathrm{e}^{-(q_1 + q_2)}), \\
l_2^{(0)} &= p_2 - \tfrac{1}{2}(a_2\, \mathrm{e}^{q_2 - q_3} + a_1\, \mathrm{e}^{q_1 - q_2} + a_{n+1}\, \mathrm{e}^{-(q_1 + q_2)}), \\
l_{n-1}^{(0)} &= p_{n-1} - \tfrac{1}{2}(a_{n-1}\, \mathrm{e}^{q_{n-1} - q_n} + a_{n-2}\, \mathrm{e}^{q_{n-2} - q_{n-3}} + a_n\, \mathrm{e}^{q_n + q_{n-1}}), \\
l_n^{(0)} &= p_n - \tfrac{1}{2}(a_{n-1}\, \mathrm{e}^{q_{n-1} - q_n} + a_n\, \mathrm{e}^{q_n + q_{n-1}}).
\end{aligned}$$

The corresponding Hamiltonian (23) acquires the following explicit form:

$$
\begin{aligned}
2H = {} & \left(p_1 + \frac{1}{2}(a_1\,\mathrm{e}^{q_1-q_2} - a_{n+1}\,\mathrm{e}^{-(q_1+q_2)})\right)^2 \\
& + \left(p_2 + \frac{1}{2}(a_2\,\mathrm{e}^{q_2-q_3} + a_1\,\mathrm{e}^{q_1-q_2} + a_{n+1}\,\mathrm{e}^{-(q_1+q_2)})\right)^2 \\
& + \sum_{i=3}^{n-2}\left(p_i + \frac{1}{2}(a_i\,\mathrm{e}^{q_i-q_{i+1}} + a_{i-1}\,\mathrm{e}^{q_{i-1}-q_i})\right)^2 \\
& + \left(p_{n-1} + \frac{1}{2}(a_{n-1}\,\mathrm{e}^{q_{n-1}-q_n} + a_{n-2}\,\mathrm{e}^{q_{n-2}-q_{n-3}} + a_n\,\mathrm{e}^{q_n+q_{n-1}})\right)^2 \\
& + \left(p_n + \frac{1}{2}(a_{n-1}\,\mathrm{e}^{q_{n-1}-q_n} + a_n\,\mathrm{e}^{q_n+q_{n-1}})\right)^2 \\
& - 2\left(\sum_{i=1}^{n-1} a_i^2\,\mathrm{e}^{2(q_i-q_{i+1})} + a_n^2\,\mathrm{e}^{2(q_n+q_{n-1})}\right. \\
& \left. + a_{n+1}^2\,\mathrm{e}^{-2(q_1+q_2)} + a_n a_{n-1}\,\mathrm{e}^{2q_{n-1}} + 2a_1 a_{n+1}\,\mathrm{e}^{-2q_1}\right) \\
& + 4\left(\sum_{i=1}^{n-1} c_i\,\mathrm{e}^{q_i-q_{i+1}} + c_n\,\mathrm{e}^{q_n+q_{n-1}} + c_{n+1}\,\mathrm{e}^{-(q_1+q_2)}\right).
\end{aligned}
$$

5.5. *Case* $\mathfrak{g} = so(2n+1)$

Let us now consider the case when the underlying Lie algebra is $so(2n+1)$. In this case we have that generic element of the space $\mathcal{M}^{\mathrm{pr}}_{1,2}(A)$ is

$$L(\lambda) = \lambda L^{(-1)} + L^{(0)} + \lambda^{-1} L^{(1)},$$

where

$$L^{(0)} = \sum_{i=1}^{n} l_i^{(0)}(X_{i+1i+1} - X_{i+n+1i+n+1}),$$

$$L^{(-1)} = \sum_{i=1}^{n-1} l_i^{(-1)}(X_{i+1i+2} - X_{i+n+2i+n+1}) + l_n^{(-1)}(X_{n+11} - X_{12n+1}) + l_{n+1}^{(-1)}(X_{n+23} - X_{n+32}),$$

$$L^{(1)} = \sum_{i=1}^{n-1} l_i^{(1)}(X_{i+2i+1} - X_{i+n+1i+n+2}) + l_n^{(1)}(X_{1n+1} - X_{2n+11}) + l_{n+1}^{(1)}(X_{3n+2} - X_{2n+3}).$$

The Poisson bracket among the coordinate functions are written as follows:

$$
\begin{aligned}
&\{l_i^{(0)}, l_j^{(0)}\}_0 = -\big(\delta_{ji+1} a_i l_{ii+1}^{(1)} - \delta_{ji-1} a_j l_{jj+1}^{(1)}\big) + a_{n+1}(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}) l_{n+1}^{(1)}, \\
&\{l_i^{(0)}, l_j^{(1)}\}_0 = (\delta_{ij} - \delta_{ij+1}) l_j^{(1)}, \qquad \{l_i^{(0)}, l_n^{(1)}\}_0 = \delta_{in} l_n^{(1)}, \\
&\{l_i^{(0)}, l_{n+1}^{(1)}\}_0 = -(\delta_{i2} + \delta_{i1}) l_{n+1}^{(1)} \{l_i^{(1)}, l_j^{(1)}\}_0 = 0, \\
&\{l_i^{(-1)}, l_j^{(-1)}\}_0 = (\delta_{jn+1} d_{i1} - \delta_{in+1} d_{j1})\big(a_1 l_{n+1}^{(-1)} - a_{n+1} l_1^{(-1)}\big) \\
&\{l_i^{(-1)}, l_j^{(0)}\}_0 = \{l_i^{(-1)}, l_j^{(1)}\}_0 = 0.
\end{aligned}
$$

It follows from this that variables $l_i^{(-1)}$ are central and could be put equal to constants if and only if $a_1 l_{n+1}^{(-1)} = a_1 l_{n+1}^{(-1)} = 0$. Let us hereafter take such a matrix A and such a 'shift-matrix'

$L^{(-1)}$ that this condition is satisfied. In this case we put $l_i^{(-1)} \equiv c_i$ and taking into account the explicit form of simple roots for $so(2n+1)$, introduce the described change of variables in subsection 5.1:

$$l_i^{(1)} = \mathrm{e}^{q_i - q_{i+1}}, \quad i = \overline{1, n}, \qquad l_n^{(1)} = \mathrm{e}^{q_n}, \qquad l_{n+1}^{(1)} = \mathrm{e}^{-(q_1+q_2)},$$
$$l_i^{(0)} = p_i - \tfrac{1}{2}(a_i\,\mathrm{e}^{q_i-q_{i+1}} + a_{i-1}\,\mathrm{e}^{q_{i-1}-q_i}), \qquad i = \overline{3, n-1},$$
$$l_1^{(0)} = p_1 - \tfrac{1}{2}(a_1\,\mathrm{e}^{q_1-q_2} - a_{n+1}\,\mathrm{e}^{-(q_1+q_2)}),$$
$$l_2^{(0)} = p_2 - \tfrac{1}{2}(a_2\,\mathrm{e}^{q_2-q_3} + a_1\,\mathrm{e}^{q_1-q_2} + a_{n+1}\,\mathrm{e}^{-(q_1+q_2)}),$$
$$l_n^{(0)} = p_n - \tfrac{1}{2}(a_{n-1}\,\mathrm{e}^{q_{n-1}-q_n} + a_n\,\mathrm{e}^{q_n}).$$

The corresponding Hamiltonian (23) has the following explicit form:

$$\begin{aligned}2H = {}& \left(p_1 + \frac{1}{2}(a_1\,\mathrm{e}^{q_1-q_2} - a_{n+1}\,\mathrm{e}^{-(q_1+q_2)})\right)^2 + \left(p_2 + \frac{1}{2}(a_2\,\mathrm{e}^{q_2-q_3} + a_1\,\mathrm{e}^{q_1-q_2} + a_{n+1}\,\mathrm{e}^{-(q_1+q_2)})\right)^2 \\ & + \sum_{i=3}^{n-1}\left(p_i + \frac{1}{2}(a_i\,\mathrm{e}^{q_i-q_{i+1}} + a_{i-1}\,\mathrm{e}^{q_{i-1}-q_i})\right)^2 + \left(p_n + \frac{1}{2}(a_{n-1}\,\mathrm{e}^{q_{n-1}-q_n} + a_n\,\mathrm{e}^{q_n})\right)^2 \\ & - 2\left(\sum_{i=1}^{n-1} a_i^2\,\mathrm{e}^{2(q_i-q_{i+1})} + a_n^2\,\mathrm{e}^{2(q_n+q_{n-1})} + a_{n+1}^2\,\mathrm{e}^{-2(q_1+q_2)} + 2a_1a_{n+1}\,\mathrm{e}^{-2q_1}\right) \\ & + 4\left(\sum_{i=1}^{n-1} c_i\,\mathrm{e}^{q_i-q_{i+1}} + c_n\,\mathrm{e}^{q_n} + c_{n+1}\,\mathrm{e}^{-(q_1+q_2)}\right).\end{aligned}$$

5.6. *Case* $\mathfrak{g} = sp(n)$

Let us now consider the case when the underlying Lie algebra is $sp(n)$. This is in a certain sense the most simple case because deformation matrix A has the most simple form for $sp(n)$ with $a_n = a_{n+1} = 0$. The generic element of the space $\mathcal{M}_{1,2}^{\mathrm{pr}}(A)$ is written as follows:

$$L(\lambda) = \lambda L^{(-1)} + L^{(0)} + \lambda^{-1}L^{(1)},$$

where

$$L^{(0)} = \sum_{i=1}^{n} l_i^{(0)}(X_{ii} - X_{i+n\,i+n}),$$
$$L^{(-1)} = \sum_{i=1}^{n-1} l_i^{(-1)}(X_{i\,i+1} - X_{i+n+1\,i+n}) + l_n^{(-1)}X_{n\,2n} + l_{n+1}^{(-1)}X_{n+1\,1}$$
$$L^{(1)} = \sum_{i=1}^{n-1} l_i^{(1)}(X_{i+1\,i} - X_{i+n\,i+n+1}) + l_n^{(1)}X_{2n\,n} + l_{n+1}^{(1)}X_{1\,n+1}.$$

The Poisson bracket among the coordinate functions has the form:

$$\{l_i^{(0)}, l_j^{(0)}\}_0 = -(\delta_{j\,i+1}a_i l_{i\,i+1}^{(1)} - \delta_{j\,i-1}a_j l_{j\,j+1}^{(1)}), \qquad \{l_i^{(0)}, l_j^{(1)}\}_0 = (\delta_{ij} - \delta_{i\,j+1})l_j^{(1)},$$
$$\{l_i^{(0)}, l_n^{(1)}\}_0 = 2\delta_{in}l_n^{(1)}, \qquad \{l_i^{(0)}, l_{n+1}^{(1)}\}_0 = -2\delta_{i1}l_{n+1}^{(1)}, \qquad \{l_i^{(1)}, l_j^{(1)}\}_0 = 0,$$
$$\{l_i^{(-1)}, l_j^{(-1)}\}_0 = \{l_i^{(-1)}, l_j^{(0)}\}_0 = \{l_i^{(-1)}, l_j^{(1)}\}_0 = 0.$$

It follows from this that the variables $l_i^{(-1)}$ are central for this bracket. Hence, we may put $l_i^{(-1)} \equiv c_i$ and introduce the described change of variables in the subsection 5.1:

$$l_i^{(1)} = e^{q_i - q_{i+1}}, \qquad i = 1, n-1, \qquad l_n^{(1)} = e^{2q_n}, \qquad l_{n+1}^{(1)} = e^{-2q_1},$$
$$l_i^{(0)} = p_i - \tfrac{1}{2}(a_i\, e^{q_i - q_{i+1}} + a_{i-1}\, e^{q_{i-1} - q_i}), \qquad i = 2, n-1,$$
$$l_1^{(0)} = p_1 - \tfrac{1}{2} a_1\, e^{q_1 - q_2}, \qquad l_n^{(0)} = p_n - \tfrac{1}{2} a_{n-1}\, e^{q_{n-1} - q_n}.$$

The corresponding Hamiltonian (23) acquires the following explicit form:

$$H = \frac{1}{2}\left(p_1 + \frac{1}{2} a_1\, e^{q_1 - q_2}\right)^2 + \frac{1}{2}\sum_{i=2}^{n-1}\left(p_i + \frac{1}{2}(a_i\, e^{q_i - q_{i+1}} + a_{i-1}\, e^{q_{i-1} - q_i})\right)^2$$
$$+ \frac{1}{2}\left(p_n + \frac{1}{2} a_{n-1}\, e^{q_{n-1} - q_n}\right)^2$$
$$- \sum_{i=1}^{n-1} a_i^2\, e^{2(q_i - q_{i+1})} + 2\left(\sum_{i=1}^{n-1} c_i\, e^{q_i - q_{i+1}} + c_n\, e^{2q_n} + c_{n+1}\, e^{-2q_1}\right).$$

5.7. Few remarks on the spectral curve

At the end of the paper we want to make several comments on the spectral curve for all of the integrable systems described in this paper. We will show how to modify the usual definition of a spectral curve in our 'deformed' situation.

The following proposition holds true:

Proposition 5.3. *Let $L(\lambda) \in \mathcal{M}_{s,p}^{\mathrm{pr}}(A)$ satisfies the 'deformed' Lax equations (17). Then the spectral curve for the 'deformed' Lax equations (17) coincide with the following curve:*

$$R_{s,p}(\lambda, \mu) \equiv \det(L(\lambda) - \mu A(\lambda)) = 0. \tag{27}$$

Proof. Presenting the curve $R_{s,p}(\lambda, \mu)$ in the form: $R_{s,p}(\lambda, \mu) = \det A(\lambda) \det(A^{-1}(\lambda) L(\lambda) - \mu 1) = 0$, we see that its coefficients are proportional to the coefficients of the characteristic polynomial of the matrix $A^{-1}(\lambda)L(\lambda)$. On the other hand, due to the Newton's identities they are independent functions of the conservative quantities of 'deformed' Lax equations (17) —integrals $I_n^k(L(\lambda))$. That proves the proposition. □

Remark 7. Note that the algebraic curve $R_{s,p}$ differs from the one that could be obtained with the help of the Kostant–Adler–Symes scheme and ordinary loop algebras [5]. In particular, the curve $R_{s,p}$ is not hyperelliptic.

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